Minnesota State High School Mathematics League

2020-21 Meet 4, Individual Event A

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 15 minutes for this event.

NO CALCULATORS are allowed on this event.

$k =$ ________________ 1. If $f(x) = \dfrac{k\sqrt{x+2}}{\sqrt{(3x-2)^3}}$ and $f(2) = 20$, what is the value of k ?

________________ 2. Determine the <u>number</u> of **real** solutions of this equation: $\sqrt{x+4} + \sqrt{x-4} = 2$.

________________ 3. Find the <u>sum</u> of all the **real** solutions of this equation: $x^4 - 4x^2 + 12x - 9 = 0$.

________________ 4. What is the <u>sum</u> of all the **integers** satisfying $\sqrt{n} + \dfrac{2}{\sqrt{n}} < 4$?

Name: ______________________________ Team: ______________________________

Minnesota State High School Mathematics League

2020-21 Meet 4, Individual Event A

SOLUTIONS

NO CALCULATORS are allowed on this event.

$\boxed{80}$

1. If $f(x)=\dfrac{k\sqrt{x+2}}{\sqrt{(3x-2)^3}}$ and $f(2)=20$, what is the value of k?

$$f(2)=\frac{k\sqrt{2+2}}{\left(\sqrt{3(2)-2}\right)^3}=\frac{2k}{8}=\frac{k}{4}=20\Rightarrow k=80.$$

$\boxed{0}$

2. Determine the <u>number</u> of **real** solutions of this equation: $\sqrt{x+4}+\sqrt{x-4}=2$.

$\sqrt{x+4}=2-\sqrt{x-4}\Rightarrow x+4=4-4\sqrt{x-4}+x-4\Rightarrow -1=\sqrt{x-4}$.
But this equation has no real solutions.

$\boxed{-2}$

3. Find the <u>sum</u> of all the **real** solutions of this equation: $x^4-4x^2+12x-9=0$.

$x^4-4x^2+12x-9=x^4-\left(4x^2-12x+9\right)=x^4-(2x-3)^2=\left(x^2-(2x-3)\right)\left(x^2+(2x-3)\right)=$
$\left(x^2-2x+3\right)\left(x^2+2x-3\right)=0$. The first factor does not factor and has **nonreal** zeros $1\pm\sqrt{2}i$.
The second factors into $(x+3)(x-1)$ and has **real** zeros -3 and 1. Their sum is -2.

$\boxed{66}$

4. What is the <u>sum</u> of all the **integers** satisfying $\sqrt{n}+\dfrac{2}{\sqrt{n}}<4$?

Let $a=\sqrt{n}$. *Then* $a+\dfrac{2}{a}<4\Rightarrow a^2-4a+2<0\Rightarrow 2-\sqrt{2}<a<2+\sqrt{2}$. *Therefore,*
$2-\sqrt{2}<\sqrt{n}<2+\sqrt{2}$. *Since all terms are positive,* $\left(2-\sqrt{2}\right)^2<n<\left(2+\sqrt{2}\right)^2\Rightarrow$
$6-4\sqrt{2}<n<6+4\sqrt{2}$. *So* $n\in\{1,\ 2,\ 3,\ \ldots,\ 11\}$ *and the sum of these integers is* $\dfrac{11\cdot 12}{2}=66$.

Minnesota State High School Mathematics League

2020-21 Meet 4, Individual Event B

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 15 minutes for this event.

$r =$ ______________

1. What is the radius of a sphere whose surface area is 256π?

______________ *cu.ft.*

2. A solid concrete porch consists of two steps and a top landing. The steps have a tread of 12 inches, a rise of 8 inches, and are 6 feet wide. The landing is 3 feet by 6 feet. A side view is shown in *Figure 2*. Determine how much concrete, in cubic feet, was used in its construction.

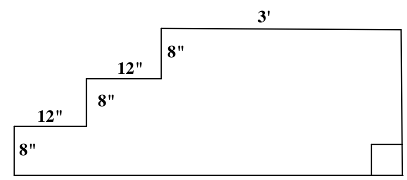

Figure 2

$r =$ ______________ *cm.*

3. A point P is 10cm. from the center of a circle. A secant through P intersects the circle at A and B so that the external segment $\overline{PA}$ is 7cm. and the internal segment $\overline{AB}$ is 5cm. Determine the radius of the circle.

$p+q=$ ______________

4. As shown in *Figure 4,* two solid steel balls, one 6 inches in diameter and the other 4 inches in diameter, are placed in a cylindrical jar 9 inches in diameter. The minimum volume of water that needs to be poured into the jar to cover both balls can be written as $\dfrac{p\pi}{q}$ cu.in., where p and q are relatively prime integers. Determine the value of $p+q$.

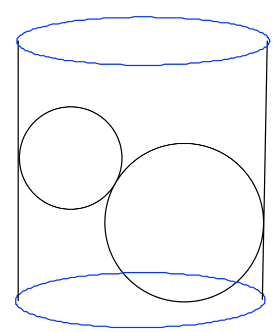

Figure 4

Name: ______________________________ Team: ______________________________

Minnesota State High School Mathematics League

2020-21 Meet 4, Individual Event B

SOLUTIONS

$\boxed{8}$

1. What is the radius of a sphere whose surface area is 256π?

$$4\pi r^2 = 256\pi \Rightarrow r^2 = 64 \Rightarrow r = 8.$$

$\boxed{48\ cu.ft.}$

2. A solid concrete porch consists of two steps and a top landing. The steps have a tread of 12 inches, a rise of 8 inches, and are 6 feet wide. The landing is 3 feet by 6 feet. A side view is shown in *Figure 2*. Determine how much concrete, in cubic feet, was used in its construction.

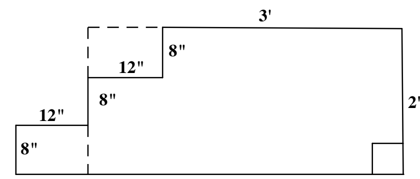

Figure 2.1

Figure 2.1 shows removing the bottom step and placing it on top of the second step, forming a $2' \times 4'$ *rectangle. Therefore, the volume is* $2 \cdot 4 \cdot 6 = 48\ cu.ft.$

$\boxed{4\ cm.}$

3. A point P is 10cm. from the center of a circle. A secant through P intersects the circle at A and B so that the external segment $\overline{PA}$ is 7cm. and the internal segment $\overline{AB}$ is 5cm. Determine the radius of the circle.

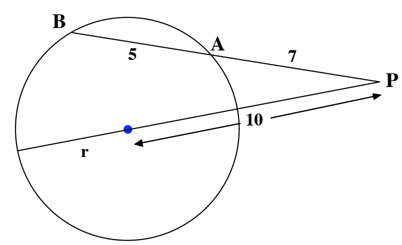

Figure 3.1

Let $r =$ *the radius of the circle. Then* $7(7+5) = (10-r)(10+r) \Rightarrow$

$84 = 100 - r^2 \Rightarrow r^2 = 16 \Rightarrow r = 4.$

$\boxed{349}$

4. As shown in *Figure 4*, two solid steel balls, one 6 inches in diameter and the other 4 inches in diameter, are placed in a cylindrical jar 9 inches in diameter. The minimum volume of water that needs to be poured into the jar to cover both balls can be written as $\dfrac{p\pi}{q}$ cu.in., where p and q are relatively prime integers. Determine the value of $p+q$.

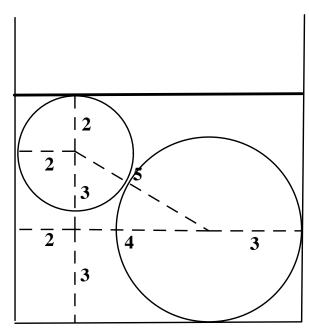

Figure 4.1

Figure 4.1 represents a vertical plane that slices through the centers of the two spheres. The distance between the two centers is $5 = 2+3$. *The horizontal distance between the two centers must be* $4 = 9-2-3$. *By the Pythagorean Theorem, the vertical distance between the two centers is* 3. *Therefore, the water needs to come to a height of* $8 = 3+3+2$. *So the volume of the water is* $(4.5)^2 \cdot \pi \cdot 8 - \frac{4}{3}\pi(3^3) - \frac{4}{3}\pi(2^3) = \frac{346\pi}{3}$.

Minnesota State High School Mathematics League

2020-21 Meet 4, Individual Event C

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 15 minutes for this event.

$c =$ ______________

1. If $32,\ a,\ b,\ c,\ 162$ form a geometric progression, what is the value of c ?

____

2. S_1 is an arithmetic sequence whose first term is 0. S_2 is a geometric sequence whose first term is not 0 and whose common ratio is not 0. S_3 is formed by adding corresponding terms of S_1 and S_2. If the first few terms of S_3 are $1,\ 1,\ 2,\ \ldots$, what is the sum of the first six terms of S_3 ?

____

3. Let $f(x)$ be a function such that $x \cdot f\left(x^2\right) + f(4x-3) = 8x$ for all integers x. Determine the value of $f(-7)$.

____

4. In the triangle in *Figure 4*, starting at any letter "I" and continually moving left, right or down to an adjacent letter, how many ways can you form the word "INTEGER"?

```
            I
          I N I
        I N T N I
      I N T E T N I
    I N T E G E T N I
  I N T E G E G E T N I
I N T E G E R E G E T N I
```

Figure 4

Name: ______________________ Team: ______________________

Minnesota State High School Mathematics League

2020-21 Meet 4, Individual Event C

SOLUTIONS

1. If $32, a, b, c, 162$ form a geometric progression, what is the value of c?

108

$b=\sqrt{32\cdot 162}=\sqrt{16\cdot 2\cdot 2\cdot 81}=4\cdot 2\cdot 9.\ c=\sqrt{b\cdot 162}=\sqrt{4\cdot 2\cdot 9\cdot 2\cdot 81}=2\cdot 2\cdot 3\cdot 9=108.$

2. S_1 is an arithmetic sequence whose first term is 0. S_2 is a geometric sequence whose first term is not 0 and whose common ratio is not 0. S_3 is formed by adding corresponding terms of S_1 and S_2. If the first few terms of S_3 are $1,\ 1,\ 2,\ \ldots$, what is the sum of the first six terms of S_3?

48

The first three terms of S_1 are $0,\ d,\ 2d$ and the first three terms of S_2 are $a,\ ar,\ ar^2$. Therefore, $0+a=1,\ d+ar=1,$ and $2d+ar^2=2$. Solving yields $a=1,\ r=2,$ and $d=-1$. The sum of the first six terms of $S_1=\frac{6}{2}(0+5(-1))=-15$ while the sum of the first six terms of $S_2=\frac{1\left(2^6-1\right)}{2-1}=63$. So the sum of the first six terms of $S_3=63-15=48$.

3. Let $f(x)$ be a function such that $x\cdot f\left(x^2\right)+f\left(4x-3\right)=8x$ for all integers x. Determine the value of $f(-7)$.

−4

To find $f(1)$, let $x=1: 1\cdot f(1)+f(1)=8\Rightarrow f(1)=4$. To find $f(-7)$, let $x=-1$:
$-1\cdot f(1)+f(-7)=-8\Rightarrow -4+f(-7)=-8\Rightarrow f(-7)=-4.$

127

4. In the triangle in *Figure 4*, starting at any letter "I" and continually moving left, right or down to an adjacent letter, how many ways can you form the word "INTEGER"?

Figure 4.1

						I
					I	N
				I	N	T
			I	N	T	E
		I	N	T	E	G
	I	N	T	E	G	E
I	N	T	E	G	E	R

All your words must end at "R" in the bottom row, so start there and spell the word backwards: "REGETNI". Consider the left "half" the triangle as in Figure 4.1. Starting at "R", each time you go either up or left to get the next six letters. So in the left "half" of the triangle there are 2^6 ways of spelling "REGETNI". There are also 2^6 ways of spelling "REGETNI" in the right "half" of the triangle. But you have counted the middle column twice. So there are $2\left(2^6\right)-1=127$ ways of spelling "INTEGER".

Minnesota State High School Mathematics League

2020-21 Meet 4, Individual Event D

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 15 minutes for this event.

NO CALCULATORS are allowed on this event.

$m+n=$ __________

1. The coordinates of the vertex of the graph of $y=x^2+12x+30$ are $(m,\ n)$. Determine the value of $m+n$.

____

2. Consider the graph of $y=x^2+7x+c$, where c is an integer. How many values of c will guarantee that the parabola has a positive y-intercept and two distinct *x*-intercepts?

$a+b+c+d=$ __________

3. A hyperbola, whose major axis is parallel with the *x*-axis, has asymptotes $y=x+4$ and $y=2-x$. The length of the minor axis is 4. The coordinates of the two foci are $(a,\ b)$ and $(c,\ d)$. Determine the value of $a+b+c+d$.

$a+b=$ __________

4. Parabola P_1 has directrix $y=-4$ and focus at $(-2,\ 4)$. Parabola P_2 has directrix $y=22$ and focus at $(-2,\ 14)$. These two foci are the foci of an ellipse, and the vertices of these parabolas are two of the vertices of the ellipse. The length of the minor (horizontal) axis of the ellipse can be written as $a\sqrt{b}$, where b is square-free. Determine the value of $a+b$.

Name: ____________________ Team: ____________________

Minnesota State High School Mathematics League

2020-21 Meet 4, Individual Event D

SOLUTIONS

NO CALCULATORS are allowed on this event.

-12

1. The coordinates of the vertex of the graph of $y = x^2 + 12x + 30$ are $(m,\ n)$. Determine the value of $m + n$.

$y = x^2 + 12x + \left(\frac{12}{2}\right)^2 + 30 - \left(\frac{12}{2}\right)^2 = (x+6)^2 - 6$. *Vertex is* $(-6, -6)$.

12

2. Consider the graph of $y = x^2 + 7x + c$, where c is an integer. How many values of c will guarantee that the parabola has a positive y-intercept and two distinct x-intercepts?

For the parabola to have a positive y-intercept, $c > 0$. *For the parabola to have two distinct x-intercepts,* $(7)^2 - 4 \cdot 1 \cdot c > 0 \Rightarrow 49 > 4c \Rightarrow c < 12\frac{1}{4}$. *Therefore, there are 12 integer values for c.*

4

3. A hyperbola, whose major axis is parallel with the x-axis, has asymptotes $y = x + 4$ and $y = 2 - x$. The length of the minor axis is 4. The coordinates of the two foci are $(a,\ b)$ and $(c,\ d)$. Determine the value of $a + b + c + d$.

The asymptotes intersect at $(-1,\ 3)$, *the center of the hyperbola. Since the positive slope of one asymptote is* 1, *and* $2b = 4$, $a = b = 2$. $c^2 = a^2 + b^2 = 8$. *Therefore,* $c = \sqrt{8}$ *and the foci are* $\left(-1 + 2\sqrt{2},\ 3\right)$ and $\left(-1 - 2\sqrt{2},\ 3\right)$. $a + b + c + d = -2 + 6 = 4$.

18

4. Parabola P_1 has directrix $y = -4$ and focus at $(-2,\ 4)$. Parabola P_2 has directrix $y = 22$ and focus at $(-2,\ 14)$. These two foci are the foci of an ellipse, and the vertices of these parabolas are two of the vertices of the ellipse. The length of the minor (horizontal) axis of the ellipse can be written as $a\sqrt{b}$, where b is square-free. Determine the value of $a + b$.

The vertex of P_1 *is* $(-2,\ 0)$. *The vertex of* P_2 *is* $(-2,\ 18)$. *The center of the ellipse is midway between these two points at* $(-2,\ 9)$. *Therefore,* $2b = 18$ *and* $2c = 10$. *In this ellipse* $a^2 = b^2 - c^2 = 81 - 25 = 56 \Rightarrow a = 2\sqrt{14} \Rightarrow 2a = 4\sqrt{14}$.

Minnesota State High School Mathematics League

2020-21 Meet 4, Team Event

Each question is worth 4 points. Team members may cooperate in any way, but at the end of 30 minutes, submit only one set of answers. Place your answer to each question on the line provided.

$p+q=$ ______________

1. If $a-b=5$ and $\sqrt[3]{a}-\sqrt[3]{b}=2$, $a\cdot b$ can be written as $\frac{p}{q}$, where p and q are relatively prime integers. Determine the value of $p+q$.

____

2. Square $ABCD$ is inscribed in a circle. Point E is on the arc $\overset{\frown}{AD}$ such that $ABCDE$ form a pentagon. If $AB=7$, determine the value of $(AE)^2+(BE)^2+(CE)^2+(DE)^2$.

$a+b=$ ______________

3. In a circle, centered at O, two perpendicular chords $\overline{AB}$ and $\overline{CD}$ intersect at point P, with $AP=2$, $CP=3$, and $BP=6$. Diameter $\overline{EF}$ also goes through P, with $FP>PE$. The length of $\overline{FP}$ can be written as $\frac{\sqrt{a}+\sqrt{b}}{2}$. Determine the value of $a+b$.

____

4. **How many** terms in the expansion of $\left(\sqrt[3]{5}+\sqrt[4]{7}\right)^{144}$ are integers?

$k=$ ______________

5. $f(x)=\log\left(\frac{1+x}{1-x}\right)$ and $g(x)=f\left(\frac{3x+x^3}{1+3x^2}\right)$. Then $g(x)$ can be simplified so that $g(x)=k\cdot f(x)$. Determine the value of k.

$a+b=$ ______________

6. An ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, with $a>b$, has $\overline{MN}$ as its major axis and $\overline{PQ}$ as its minor axis and point O as its center. The $\tan(\angle PMQ)=2\sqrt{6}$. If $a=6$, the distance between the foci of the ellipse can be written as $a\sqrt{b}$. Determine the value of $a+b$.

Team: ______________________________

2020-21 Meet 4, Team Event

SOLUTIONS (page 1)

$\boxed{7}$

1. If $a-b=5$ and $\sqrt[3]{a}-\sqrt[3]{b}=2$, $a\cdot b$ can be written as $\dfrac{p}{q}$, where p and q are relatively prime integers. Determine the value of $p+q$.

$\boxed{196}$

2. Square $ABCD$ is inscribed in a circle. Point E is on the arc $\overset{\frown}{AD}$ such that $ABCDE$ form a pentagon. If $AB=7$, determine the value of $(AE)^2+(BE)^2+(CE)^2+(DE)^2$.

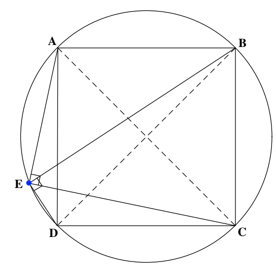

Figure 2.1

$\boxed{82}$

3. In a circle, centered at O, two perpendicular chords $\overline{AB}$ and $\overline{CD}$ intersect at point P, with $AP=2$, $CP=3$, and $BP=6$. Diameter $\overline{EF}$ also goes through P, with $FP>PE$. The length of $\overline{FP}$ can be written as $\dfrac{\sqrt{a}+\sqrt{b}}{2}$. Determine the value of $a+b$.

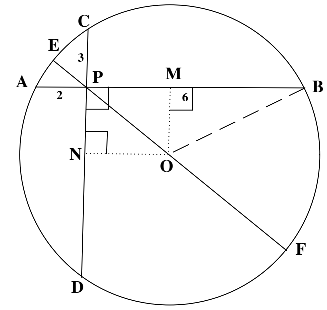

Figure 3.1

$\boxed{13}$

4. **How many** terms in the expansion of $\left(\sqrt[3]{5}+\sqrt[4]{7}\right)^{144}$ are integers?

$\boxed{3}$

5. $f(x)=\log\left(\dfrac{1+x}{1-x}\right)$ and $g(x)=f\left(\dfrac{3x+x^3}{1+3x^2}\right)$. Then $g(x)$ can be simplified so that $g(x)=k\cdot f(x)$. Determine the value of *k*.

$\boxed{7}$

6. An ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, with $a>b$, has $\overline{MN}$ as its major axis and $\overline{PQ}$ as its minor axis and point O as its center. The $\tan(\angle PMQ)=2\sqrt{6}$. If $a=6$, the distance between the foci of the ellipse can be written as $a\sqrt{b}$. Determine the value of $a+b$.

Minnesota State High School Mathematics League

2020-21 Meet 4, Team Event

SOLUTIONS (page 2)

1. *Cubing both sides of the second equation yields* $a-3\sqrt[3]{a^2}\sqrt[3]{b}+3\sqrt[3]{a}\sqrt[3]{b^2}-b=8 \Rightarrow 3\sqrt[3]{a}\sqrt[3]{b}\left(\sqrt[3]{b}-\sqrt[3]{a}\right)=8+b-a \Rightarrow$ $3\sqrt[3]{ab}(-2)=8-5 \Rightarrow \sqrt[3]{ab}=\frac{-1}{2} \Rightarrow ab=\frac{-1}{8}$. *Therefore,* $p=-1$ *and* $q=8$, *so* $p+q=7$.

2. *Draw diameters* $\overline{AC}$ *and* $\overline{BD}$ *as in Figure 2.1. By Thales' Theorem,* $\triangle AEC$ *is a right triangle with* $(AE)^2+(CE)^2=(AC)^2=\left(7\sqrt{2}\right)^2=98$. *Likewise,* $\triangle BED$ *is a right triangle with* $(BE)^2+(DE)^2=(BD)^2=\left(7\sqrt{2}\right)^2=98$. *So* $(AE)^2+(BE)^2+(CE)^2+(DE)^2=98+98=196$.

3. $(AP)(BP)=(CP)(DP) \Rightarrow DP=4$. *Let* M *and* N *be the midpoints of* $\overline{AB}$ *and* $\overline{CD}$, *respectively. Draw* $\overline{OM}$ *and* $\overline{ON}$. *They will form right angles.* $MO=PN=0.5$, $MB=4$ *and* $MP=2$. *Therefore,* $OP=\sqrt{\left(\tfrac{1}{2}\right)^2+2^2}=\frac{\sqrt{17}}{2}$. $OF=OB=\sqrt{\left(\tfrac{1}{2}\right)^2+4^2}=\frac{\sqrt{65}}{2}$. *Therefore,* $FP=OP+OF=\frac{\sqrt{17}+\sqrt{65}}{2}$.

4. $\binom{144}{k}\left(5^{\frac{1}{3}}\right)^{144-k}\left(7^{\frac{1}{4}}\right)^{k}=\binom{144}{k}\left(5^{\frac{144-k}{3}}\right)\left(7^{\frac{k}{4}}\right)$. *For* $0\le k\le 144$, $\binom{144}{k}$ *is always an integer,* $\left(5^{\frac{144-k}{3}}\right)$ *is an integer if* $144-k$ *is divisible by* 3, *and* $\left(7^{\frac{k}{4}}\right)$ *is an integer if* k *is divisible by 4. Since* 144 *is divisible by* 3, k *must be divisible by* 3 *and* 4. *The multiples of* 12 *satisfy this condition and there are* 13 *in the interval* $[0,\ 144]$.

5. $$g(x)=\log\left(\frac{1+\dfrac{3x+x^3}{1+3x^2}}{1-\dfrac{3x+x^3}{1+3x^2}}\right)=\log\left(\frac{\dfrac{1+3x^2+3x+x^3}{1+3x^2}}{\dfrac{1+3x^2-\left(3x+x^3\right)}{1+3x^2}}\right)=\log\frac{x^3+3x^2+3x+1}{1-3x+3x^2-x^3}=\log\frac{(x+1)^3}{(1-x)^3}=\log\left(\frac{1+x}{1-x}\right)^3=$$ $$3\cdot\log\left(\frac{1+x}{1-x}\right)=3\cdot f(x).$$

6. *In isosceles* $\triangle PMQ$, *let* $m\angle PMQ=\theta$. *Then* $\triangle PMO$ *is a right triangle with* $MO=a$, $PO=b$, $\tan\angle PMO=\frac{b}{a}$, *and* $m\angle PMO=\frac{\theta}{2}$. *From the Double Angle Theorem,* $\tan\theta=\frac{2\tan\left(\frac{\theta}{2}\right)}{1-\tan^2\left(\frac{\theta}{2}\right)}$. *Therefore, letting* $\tan\theta=2\sqrt{6}$, $2\sqrt{6}-2\sqrt{6}\tan^2\left(\frac{\theta}{2}\right)=2\tan\left(\frac{\theta}{2}\right) \Rightarrow 0=\sqrt{6}\tan^2\left(\frac{\theta}{2}\right)+\tan\left(\frac{\theta}{2}\right)-\sqrt{6} \Rightarrow \tan\left(\frac{\theta}{2}\right)=\frac{\sqrt{6}}{3}$. *Therefore,* $\frac{b}{6}=\frac{\sqrt{6}}{3} \Rightarrow b=2\sqrt{6}$. *In an ellipse,* $c^2=a^2-b^2 \Rightarrow c=\sqrt{36-24}=2\sqrt{3}$. *So the distance between the foci is* $4\sqrt{3}$.